WIGNER TRANSFORM AND QUANTUM-LIKE CORRECTIONS FOR CHARGED-PARTICLE BEAM TRANSPORT

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It is shown that charged-particle beam transport in the paraxial approximation can be effectively described with a quantum-like picture in *semiclassical approximation*. In particular, the classical Liouville equation can be suitably replaced by a von Neuman-like equation. Relevant remarks concerning the standard classical description of the beam transport are given.

It is well known that *thermal spreading among the electronic rays* is a typical effect that takes place in charged-particle beam transport in free space.[1] In 2-D case, we denote with x and z the transverse and beam propagation coordinates, respectively. By using a statistical description, one can introduce with the second-order moments: $\sigma_x(z) = \langle x^2\rangle^{1/2}$, $\sigma_p(z) = \langle p^2\rangle^{1/2}$, and $\sigma_{xp} = \langle xp\rangle$ ($p \equiv dx/dz$ being a dimensionless single-particle linear momentum conjugate of x, where z plays the role of a time-like variable) the following diffusion coefficient called the emittance,[1] $\epsilon = 2\left[\langle x^2\rangle\langle p^2\rangle - \langle xp\rangle^2\right]^{1/2}$, which for linear lens and in free space is an invariant. From this expression, in particular, we have $\sigma_x\sigma_p \geq \epsilon/2$, which represents a sort of *uncertainty relation* even if the particle beam is a *classical system*. It is easy to prove that[1] $(\epsilon/2) = v_{\rm th}\sigma_0/c$, where $v_{\rm th}$ is the *thermal velocity* of the system (we assume $v_{\rm th}/c \ll 1$), and $\sigma_0 \equiv \langle x^2\rangle^{1/2}_{z=0}$.

Thus, for finite temperature the determination of an electronic ray at the arbitrary x-position of the transverse plane given at each z is affected by an intrinsic uncertainty that cannot be reduced to zero. For a finite emittance, we need to assign a probability, say $P_x(x,z)$, (in principle, positive and finite) of finding an electronic ray at the transverse location x in the plane for given z. To this end, we introduce the phase-space density distribution $\rho(\overline{x}, p, \overline{z})$, which obeys to the following classical Liouville equation for the electronic rays,

$$\frac{\partial\rho}{\partial\overline{z}} + p\frac{\partial\rho}{\partial\overline{x}} - \left(\frac{\partial\overline{U}}{\partial\overline{x}}\right)\frac{\partial\rho}{\partial p} = 0, \quad \overline{z} \equiv \frac{z}{2\sigma_0}\,, \quad \overline{x} \equiv \frac{x}{2\sigma_0}\,,$$

$\overline{U} = \overline{U}(\overline{x}, \overline{z})$ being an effective potential acting on the system. Since for finite emittance the indistinguishability among two or more rays due to the thermal

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spreading is of the order of $\eta \equiv \epsilon/2\sigma_0 = v_{\rm th}/c \ll 1$, $\partial\overline{U}/\partial\overline{x}$ can be conveniently replaced by a symmetrized Schwarz-like finite difference ratio

$$\frac{\partial\overline{U}}{\partial x}\frac{\partial}{\partial p} \approx \frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{\eta}\frac{\partial}{\partial p}\,.$$

This *transition* based on physical arguments is partially a change of partial differential equation to a differential-difference equation, which may be considered as ansatz of a *deformation* of the Liouville equation. Furthermore, given the smallness of η, multiplying by the imaginary unit i both numerator and denominator of the last term of the l.h.s., we have

$$\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{i\eta}\, i\frac{\partial}{\partial p} \approx \frac{1}{i\eta}\left[\overline{U}\left(\overline{x}+\frac{i\eta}{2}\frac{\partial}{\partial p}\right)-\overline{U}\left(\overline{x}-\frac{i\eta}{2}\frac{\partial}{\partial p}\right)\right].$$

Thus, going back to the old variables x and z, it finally results that the classical Liouville equation is *deformed* in the following von Neumann equation

$$\left\{\frac{\partial}{\partial z}+p\frac{\partial}{\partial x}+\frac{i}{\epsilon}\left[U\left(x+i\frac{\epsilon}{2}\frac{\partial}{\partial p}\right)-U\left(x-i\frac{\epsilon}{2}\frac{\partial}{\partial p}\right)\right]\right\}\rho_w = 0\,, \tag{1}$$

where the *deformed* distribution function $\rho_w(x,p,z)$ is a sort of Wigner-like function.[2] It is obvious, that Eq. (1) has the form of a quantum-like phase-space equation for electronic rays, where $\hbar$ and the time t are replaced by the emittance ϵ and the propagation coordinate z, respectively. However, some aspects have to be discussed.

(i). Since $\quad \overline{U}(\overline{x}+\frac{i\eta}{2}\frac{\partial}{\partial p})-\overline{U}(\overline{x}-\frac{i\eta}{2}\frac{\partial}{\partial p}) = \frac{\partial\overline{U}}{\partial\overline{x}}\; i\eta\frac{\partial}{\partial p}+O\left(\eta^3\frac{\partial^3}{\partial p^3}\right),$

the above approximation is equivalent to assume that terms $O\left(\eta^3\,\partial^3/\partial p^3\right)$ are small corrections compared to the lower-order ones, according to the paraxial approximation. Consequently, from the quantum-like point of view, approximation obtained by the above deformation plays the role analogous to the one played by *semiclassical approximation*.[3]

(ii). While $\rho(x,p,z)$ is introduced in a classical framework and it is positive definite, the function $\rho_w(x,p,z)$ is introduced in a quantum-like framework, which plays the role of an *effective* description taking into account the thermal spreading among the electronic rays. In this context, $\rho_w(x,p,z)$ cannot be used to give information within the phase-space cells with size smaller than ϵ, due to the intrinsic uncertainty exhibited by the system for finite temperatures, i.e., due to the indistinguishability among the electronic rays. Consequently, we would expect that ρ_w violates the positivity definiteness within

some phase-space regions. This means that, in analogy with quantum mechanics, $\rho_w(x,p,z)$ can be defined as *quasi distribution*, even its x- and p-projections are actually configuration-space and momentum-space distributions, respectively. Remarkably, from the above results it follows that it can exist a complex function, say $\Psi(x,z)$ such that $P_x(x,z) = \Psi(x,z)\Psi^*(x,z)$, which is connected with ρ_w by means of a Wigner-like transform (for pure states).[4] Consequently, $\Psi(x,z)$ must obey to the following Schrödinger-like equation:

$$i\epsilon(\partial\Psi/\partial z) = -(\epsilon^2/2)\partial^2\Psi/\partial x^2 + U(x,z)\Psi \quad , \tag{2}$$

which has been the starting point to construct the quantum-like approach called the thermal wave model (TWM).[5] This way, the beam as a whole is thought as a single quantum-like particle whose *diffraction-like* spreading due to the emittance (the analogous of $\hbar$) accounts for the *thermal spreading*.

Thus, we have given a sort of *Wigner-like* pictures behind the electronic ray evolution and then recovered TWM in semiclassical approximation. Consequently, solutions of (2) for Ψ in semiclassical approximation can give solutions for the *deformed* equation through Wigner transform. It is clear that for finite emittance but in the case in which, for $s \geq 3$, $(\epsilon/2)^2\,\partial^2\rho_w/\partial p^2 \gg (\epsilon/2)^s\partial^s\rho_w/\partial p^s \quad \text{for} \quad s \geq 3$, (1) and its classical couterpart formally coincide for an arbitrary (anharmonic) potential; the similarity between ρ and ρ_w does not take place for all the states. This makes evident that for an arbitrary potential and, in particular, for a linear lens (harmonic oscillator) ρ_w contains and ρ does not contain a *quantum-like effect*. Worthy noting that, in analogy with the tomography approach in quantum mechanics and quantum optics,[6] we could state that in the above quantum-like approach there is a possibility to transit from Liouville equation to an equation for a positive marginal distribution,[6] which has standard classical features.

References

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